Some remarks on the inverse eigenvalue problem for real symmetric Toeplitz matrices

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Abstract

Two theorems about the solution properties of the Toeplitz Inverse Eigenvalue Problem (ToIEP) are introduced and proved. One of them is applied to make a better starting generator and the other can be used to double the number of solutions found. These applications are tested through a short *Mathematica* program. Also an optimisation method for solving ToIEP with global convergence property is presented. The global convergence theorem is proved.

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1 Introduction

An inverse Toplitz eigenvalue problem(ToIEP) is to obtain a real vector $\mathbf{r} = [r_1, r_2, \cdots, r_n]^t$ so that the Toeplitz matrix

$$T(\mathbf{r}) = \begin{bmatrix} r_1 & r_2 & \cdots & r_{n-1} & r_n \\ r_2 & r_1 & \cdots & r_{n-2} & r_{n-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ r_{n-1} & r_{n-2} & \cdots & r_1 & r_2 \\ r_n & r_{n-1} & \cdots & r_2 & r_1 \end{bmatrix}. \tag{1}$$

has a prescribed set of real numbers $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ as its spectrum. The sovability issue has been settled by Landau[8] who proved that every set of n real numbers is the spectrum of an $n \times n$ real symmetric Toeplitz matrix. Because of its nonconstructive proof, Numerical, basically, Newton-type iteration methods are still the main tool to build up such Toeplitz matrices.

The critical task for applying Newton's method is to choose a starting point or an initial approximation properly, otherwise the iterations would either diverge or converge to a point which is not a solution. The issue for ToIEP is also mentioned by Laurie[9] and Trench[15]. In Section 2 of this paper an estimation of solutions(Theorem 1) is provided which shows the bounds of each component of a solution r. Therefore it clearly gives the guidance for choosing a starting point. A more reliable starting generator is thus produced and tested by a short Mathematica program in Section 3. Theorem 2 reveals an interesting fact, i.e. the solutions of ToIEP exist in pairs. This discovery is very helpful in searching of all possible solutions of ToIEP.

There are two categories of the iterative methods for ToIEP. One exploits the Toeplitz structure while the other[1, 6, 5, 9] does not. All these methods except Trench's do not possess a global convergence property. The Trench's one appears to be globally convergent; however, this has not been proved. In Section 4 an optimization approach with a global convergence feature is presented. The Levenberg-Marquardt(L-M) method for solving this minimization problem is applied[13, 14]. The L-M method is widely recognized as the one of the most reliable methods for nonlinear least squares problems. This optimization approach itself does not need any knowledge of the Toeplitz structure but its global convergence does depend on it.

2 Two theorems about the solutions

The first theorem gives the bounds of each component of a solution $\mathbf{r}$.

Theorem 1 *If* $\mathbf{r} = [r_1, r_2, \cdots, r_n]^t$ *is a solution of the ToIEP, then*

$$r_1 = \sigma_1/n \tag{2}$$

and

$$|r_i| \leq \sqrt{\frac{n\sigma_2 - \sigma_1^2}{2n(n-i+1)}} \quad i = 2, \cdots, n \tag{3}$$

where $\sigma_k = \sum_{i=1}^n \lambda_i^k$

Proof:
Equation 2 is obvious. To derive Inequation 3, consider c_2, the coefficient of λ^{n-2} term in the characteristic polynomial $\mathrm{Det}(\lambda I - T(\mathbf{r}))$ which should be equal to $\lambda_1\lambda_2 + \cdots + \lambda_{n-1}\lambda_n = (\sigma_1^2 - \sigma_2)/2$. calculation using Faddeev formula[7] yields

$$\begin{aligned} c_2 &= 1/2\,\mathrm{trace}(\mathbf{T}((\mathrm{trace}\mathbf{T})\mathbf{I} - \mathbf{T})) \\ &= 1/2((\mathrm{trace}\mathbf{T})^2 - \mathrm{trace}\mathbf{T}^2) \\ &= \frac{n(n-1)}{2} r_1^2 - (n-1)r_2^2 - (n-2)r_3^2 - \cdots - r_n^2 \\ &= (\sigma_1^2 - \sigma_2)/2 \end{aligned} \tag{4}$$

After applying Equation 2, we obtain

$$(n-1)r_2^2 + (n-2)r_3^2 + \cdots + r_n^2 = \frac{n\sigma_2 - \sigma_1^2}{2n} \tag{5}$$

which immediately gives Inequation 3.

It is easy to see that for the problem with standardized eigenvalues($\sigma_1 = 0, \sigma_2 = 1$) [15]

$$|r_i| \leq \frac{1}{\sqrt{2(n-i+1)}} \qquad i = 2, \cdots, n \tag{6}$$

This theorem gives a clear criterion for selecting an initial approximation when a numerical method is applied to find the solutions of the problem. Next theorem shows that the solutions of the ToIEP exist in pairs. It is helpful when we are trying to locate all possible solutions of the problem.

Theorem 2 *For a given set of real numbers* $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$, *if*

$$\mathbf{r} = [r_1, r_2, \cdots, r_n]^t$$

is a solution of the ToIEP, then

$$\tilde{\mathbf{r}} = [r_1, -r_2, \cdots, (-1)^{n-1} r_n]^t$$

is also a solution of the ToIEP

In order to prove this theorem we need the following lemma.

Lemma 1 *Let $t_{i,j}^{(k)}$ and $\tilde{t}_{i,j}^{(k)}$ be the i, j-th entries of $\mathbf{T}^k(\mathbf{r})$ and $\mathbf{T}^k(\tilde{\mathbf{r}})$ respectively, then $t_{i,j}^{(k)} = (-1)^{j-i} \tilde{t}_{i,j}^{(k)}$ for all natural integers k.*

Proof:
For $k = 1$ it is obvious. We assume that the lemma is true for k. Then for $k + 1$ we have

$$\begin{aligned} t_{i,j}^{(k+1)} &= [r_i, r_{i-1}, \cdots, r_1, \ldots, r_{n-i+1}][t_{1,j}^{(k)}, t_{2,j}^{(k)}, \cdots, t_{n,j}^{(k)}]^t \\ &= r_i t_{1,j}^{(k)} + r_{i-1} t_{2,j}^{(k)} + \cdots + r_{n-i+1} t_{n,j}^{(k)} \\ &= (-1)^{1-i} \tilde{r}_i (-1)^{j-1} \tilde{t}_{1,j}^{(k)} + (-1)^{2-i} \tilde{r}_{i-1} (-1)^{j-2} \tilde{t}_{2,j}^{(k)} + \cdots \\ &\quad + (-1)^{n-i} \tilde{r}_{n-i+1} (-1)^{j-n} \tilde{t}_{n,j}^{(k)} \\ &= (-1)^{j-i} \tilde{t}_{i,j}^{(k+1)} \end{aligned} \tag{7}$$

Therefore by induction the lemma is true for all k.
Now we can prove Theorem 2

Proof:
Lemma 1 shows that

$$\mathrm{trace}(\mathbf{T}^k(\mathbf{r})) = \mathrm{trace}(\mathbf{T}^k(\tilde{\mathbf{r}})),$$

for $k = 1, 2, \cdots, n$. Because $\mathrm{trace}(\mathbf{T}^k(\mathbf{r}))=\sigma_k=\sum_{i=1}^n \lambda_i^k$, and by the well known Newton's formula for symmetric polynomial[7], the coefficients of the characteristic polynomial of $T(\mathbf{r})$ are identical to the ones of $T(\tilde{\mathbf{r}})$. Thus the matrix $T(\tilde{\mathbf{r}})$ also has the prescribed eigenvalues $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$.

3 A *Mathematica* program

The starting generator is usually a subtle issue when applying iterative methods. Trench, Laurie and other authers have mentioned the issue for solving Toeplitz problems[8,14]. Some generators make a unified staring value for $r_2, r_3, \ldots, r_n$, e.g. $\frac{1}{2(n-1)}$, ignoring the difference among these components.

In fact from Theorem 1 it can be seen that the bounds for r_2 and r_n are differed by nearly $\sqrt{n}$ times. When n is large the ignorance will not be acceptable. The following short Mathematica program is designed for solving ToEIP which shows how the results of Theorem 1 are used to initiate the subroutine **FindRoot**. The i-th component of a starting point $\mathbf{r}$ is chosen randomly between

$$\pm 0.5\sqrt{\frac{n\sigma_2 - \sigma_1^2}{2n(n-i+1)}}$$

using **Random**[], which produces a random number between 0 and 1. The algorithm used in the program is quite simple, just solve the equations obtained by equating corresponding coefficients of the characteristic polynomial of $T(\mathbf{r})$ and $P(x) = (x - \lambda_1)\cdots(x - \lambda_n)$. We test the algorithm on a problem with an extremely irregularly clustered spectral data $\{1000, 100, 99, 5, 1\}$ which was first presented by Laurie [9]. The program makes 100 tries. The program is as follows:

```
λ[1]=1000; λ[2]=100; λ[3]=99;
λ[4]=5;λ[5]=1;
s1=Sum[λ[i],{i,1,5}]; s2=Sum[λ[i]^ 2,{i,1,5}];
a=s1/5;
m={{a,b,c,d,e}, {b,a,b,c,d}, {c,b,a,b,c}, {d,c,b,a,b}, {e,d,c,b,a}};
P[x_]:=Product[(x- λ[i]),{i,1,5}]
eqs=Table[ Coefficient[Det[x*IdentityMatrix[5]-m],x,i]==
Coefficient[P[x],x,i],{i,0,3}];
start[k_]:= (Random[]-0.5)Sqrt[5 s2-s1^ 2)/(10*(6-k))];
For[i=1, i ≤ 100, i++,
Do[sol=
FindRoot[eqs,{b,start[2]} ,{c, start[3]}, ,{d, start[4]},{e, start[5]}];
Print[sol]]]
```

After 100 tries, we obtained the following 12 sets of solutions of (b, c, d, e) with $a = 241.000$:

{168.853, 212.453, 209.583, 165.547}, {-191.89, 218.846, -155.583, 159.154}, {-192.256, 218.631,-155.536, 158.369}, {168.986, 212.011, 210.26, 164.989}, {-211.225, 169.31, -166.489, 211.69}, {193.838, 217.022, 152.043, 163.978}, {210.868, 168.858, 167.156, 213.142}, {-185.502, 160.523, -224.893, 220.477}, {193.472, 217.237, 152.089, 164.763}, {186.977, 159.793, 224.821, 217.207}, {167.541, 210.216, 212.945, 170.784}, {167.399, 210.668, 212.278, 171.332}.

Actually, from Theorem 2, we have obtained 24 sets of solutions e.g. from the first solution we can obtain that one with $a = 241.000, b = -168.853, c = 212.453, d = -209.583, e = 165.547$ is also a solution for the problem. We can expect to obtain more solutions (possibly 5!=120 solutions, see [2, 6]) if we try more times.

4 An optimisation method

In the above program we convert the ToIEP to a system of polynomial equations,

$$f_i(r_2, \cdots, r_n) = c_i(r_2, \cdots, r_n) - p_i = 0, \quad i = 2, \cdots, n$$

where c_i and p_i are coefficients of the λ^{n-i} term of the characteristic polynomial of $T(r)$ with $r_1 = \sigma_1/n$ and the polynomial $P(x) = (x - \lambda_1)\cdots(x - \lambda_n)$, respectively. Then use the equation solver **FindRoot** to obtain the solutions.

We now apply the least squares optimization method to find the solutions of equations. The object function to be minimized here is

$$F(r_2, \cdots, r_n) = 1/2 \sum_{i=2}^{n} f_i^2(r_2, \cdots, r_n)$$

Obviously, if at a stage in the minimization process $F(\bar{r}_2, \cdots, \bar{r}_n) = 0$, then $\mathbf{r} = (r_1, \bar{r}_2, \cdots, \bar{r}_n)$ is a solution of the ToIEP. The well known Levenberg-Marquardt method (L-M method) is applied for solving the minimization problem. The L-M method is widely recognized as the one of the most reliable methods for nonlinear least squares problems. It works extremely well for functions without a high degree of nonlinearity [10, 11, 12]. Powell obtained a global convergence result Theorem 3 for his hybrid version of the L-M method for the case when the elements of the Jacobian are exact[13]. Interestingly, because of the special structure of the equation , when the L-M method is employed to our problem, it possesses an elegant global convergence property. This is the main reason why this method should be used here. Before the proof of the global convergence, we list Theorem 3 first.

Theorem 3 (Powell [14]) *If the functions f_i have continuous, bounded first derivatives then the* L-M *method will finish after a finite number of iterations, due to*

$$F(\mathbf{x}) < E$$

or

$$F(\mathbf{x}^{(k)}) \geq M\|\mathbf{g}^{(k)}\|_2,$$

where E and M are assigned fixed positive values before the iterations begin and $\mathbf{g}^{(k)}$ is the gradient vector of $F(x)$ at the k-th iterate $\mathbf{x} = \mathbf{x}^{(k)}$.

Let $\mathbf{x} = (r_2, \cdots, r_n)$, if $\mathbf{x}^{(0)}$ is an initial approximation to the solutions of the equations (4), then the L-M method restricts all iterates $\mathbf{x}^{(k)}$ to the set

$$S = \{\mathbf{x} : F(\mathbf{x}) \leq F(\mathbf{x}^{(0)})\}.$$

We claim that S is a compact set. As F is a continous function S must be closed, hence we only need to prove that S is bounded. Let $c = (2F(\mathbf{x}^{(0)})1/2$ Then

$$|f_2| = |(n-1)r_2^2 + (n-2)r_3^2 + \cdots + r_n^2 - \frac{n\sigma_2 - \sigma_1^2}{2n}| \leq c$$

The above inequality leads to

$$|r_i| \leq \sqrt{\frac{n\sigma_2 - \sigma_1^2 + 2nc}{2n(n-i+1)}} \quad i = 2, \cdots, n$$

Thus the set S is bounded. Because all the derivatives f_i' are polynomials on the compact set S, they must be bounded. Therefore the conditions of Theorem 3 are fulfilled and the following global convergence theorem follows.

Theorem 4 *Powells version of L-M method for solving ToIEP has a global convergence property.*

5 Conclusion

Two theorems about the solution properties of the ToIEP are introduced and proved in this paper. One of them is applied to make a better starting generator and the other can be used to double the solutions found. These applications are tested through a short *Mathematica* program. Also an optimization method with global convergence property is presented. The global convergence theorem is proved.

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