A Boundary Element Method for Infiltration from Irrigation Channels in a Soil with Impermeable Inclusions

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Abstract

This study is concerned with the effect of impermeable inclusions on the infiltration of water from irrigation channels in a homogeneous soil. An expression for the matric flux potential throughout the soil is obtained in terms of a boundary integral equation. A Green's functions for use in the boundary integral equation is derived. This Green's function is suitable for numerical calculations and is employed in the boundary integral equation to obtain numerical values for the matric flux potential for a soil with embedded impermeable inclusions of various shapes. The numerical results indicate how impermeable inclusions may be used to effectively direct the flow from irrigation channels to particular regions below the soil surface.

Contents

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1 Introduction

A number of researchers have undertaken analysis of steady infiltration into unsaturated soils. For example Philip [5], [6],Raats [7] and Batu [3] have solved steady infiltration problems from a point, line strip, and disc sources. All these authors considered infiltration through a uniform homogenous soil.

The present study is concerned with the solution of a class of infiltration problems from one or more irrigation channels into a soil with impermeable inclusions (see Figure 1). The paper is an extension of work previously developed by Azis, Clements and Lobo [1] on the use of boundary element method for steady infiltration from irrigation channels in a soil. The aim of the study is to observe how the impermeable inclusions below the soil surface influence the direction of the flow of water through the soil.

The governing equation that is solved in this study is the linearized form of the infiltration equation

$$\frac{\partial^2\Theta}{\partial X^2}+\frac{\partial^2\Theta}{\partial Z^2}=\alpha\frac{\partial\Theta}{\partial Z} \tag{1}$$

(see for example Batu [3]) where Θ is the matric flux potential throughout the soil.

A boundary integral equation formulation is used to facilitate the numerical solution of the governing differential equation and this is then used to determine the value of the matric flux potential Θ in a soil with impermeable inclusions of various shapes. The solutions obtained are relevant in assessing the influence of an impermeable inclusion in directing the flow from irrigation channels to a particular regions below the soil surface.

2 Statement of the problem

Referred to a Cartesian frame $OXYZ$ consider an isotropic homogenous soil lying in the region $Z > 0$ with OZ vertically downwards. The region contains one or two semi-circular channels and also impermeable inclusions (see Figures 1 and 2). The channel has surface area $2L$ per unit length in the OY direction where L is the reference length and the channel is filled with water.

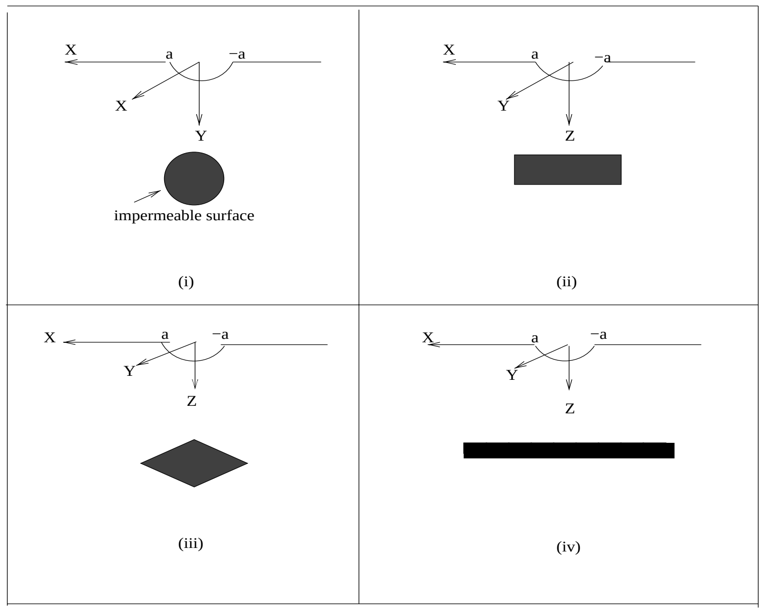

Figure 1: Illustration of the physical problem of the single channel

Impermeable inclusions of finite width and length are located in the soil in such a way that they do not intersect the boundary or other impermeable inclusions (see Figures 1 and 2). Each semicircular channel has a radius of $a = 2L/\pi$. For the case of two channels (see Figure 2) the distance between the center of the channels is taken to be $10L$. The inclusions are placed in the soil as shown in Figures 1 and 2 at a depth of $Z = \beta L$ where β is a dimensionless parameter.

The normal flow is taken to be zero over the surface boundary along $Z = 0$ outside the channel. Over the surface of the channel a uniform constant flow is specified normal to the surface of the channel.

It is required to determine the matric flux potential $\Theta(X, Z)$ and the flow throughout the soil $Z > 0$ and to observe the effect of the impermeable inclusion on this flux and flow. It is assumed that the matric flux potential Θ and the derivatives $\partial\Theta/\partial X$ and $\partial\Theta/\partial Z$ vanish as $X^2 + Z^2$ tends to infinity.

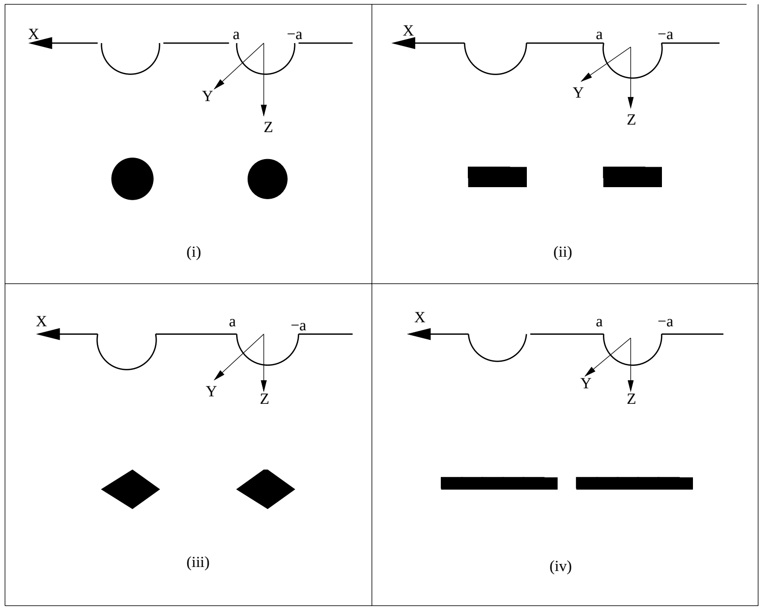

Figure 2: Illustration of the physical problem of the double channels

3 Fundamental Equations

The matric flux potential Θ is taken to be related to the hydraulic conductivity by the equation (see Gardner [4])

$$\Theta = \int_{-\infty}^{h} K(q)dq = \alpha^{-1}K(h), \tag{2}$$

with

$$K(h) = K_s \exp(\alpha h), \tag{3}$$

where h (units L) is the soil water potential and α (units L^{-1}) is an empirical constant and K_s and $K(h)$ denotes the hydraulic conductivities in saturated soil and unsaturated soil respectively.

Equation (1) is the linearized form of the steady infiltration equation with the horizontal and vertical components of the flux, given by

$$U = -\frac{\partial\Theta}{\partial X}, \quad \text{and} \quad V = \alpha\Theta - \frac{\partial\Theta}{\partial Z}, \tag{4}$$

respectively. The flux normal to a surface with outward pointing normal $\mathbf{n} = (n_1, n_2)$ is given by

$$F = -\frac{\partial\Theta}{\partial X}n_1 + (\alpha\Theta - \frac{\partial\Theta}{\partial Z})n_2. \tag{5}$$

Dimensionless variables are now defined in the form

$$\theta = \frac{1}{V_0 L}\Theta, \quad x = \frac{\alpha}{2}X, \quad z = \frac{\alpha}{2}Z,$$
$$u = \frac{2}{V_0 \alpha L}U, \quad v = \frac{2}{V_0 \alpha L}V, \quad f = \frac{2}{V_0 \alpha L}F, \tag{6}$$

where V_0 is a reference flux. In terms of these variables equations (1), (4) and (5) may be written in the dimensionless form

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial z^2} = 2\frac{\partial \theta}{\partial z}, \tag{7}$$

$$u = -\frac{\partial \theta}{\partial x}, \quad v = 2\theta - \frac{\partial \theta}{\partial z}, \tag{8}$$

$$f = -\frac{\partial \theta}{\partial x}n_1 + (2\theta - \frac{\partial \theta}{\partial z})n_2. \tag{9}$$

The transformation

$$\theta = \exp{(z)}\Psi \tag{10}$$

transforms equation (7) to

$$\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial z^2} - \Psi = 0. \tag{11}$$

Also equations (8) and (9) transform to

$$u = -\exp{(z)}\frac{\partial \Psi}{\partial x}, \quad v = \exp{(z)}(\Psi - \frac{\partial \Psi}{\partial z}), \tag{12}$$

$$f = -\exp{(z)}\left[\frac{\partial \Psi}{\partial n} - \Psi n_2\right]. \tag{13}$$

Hence

$$\frac{\partial \Psi}{\partial n} = \Psi n_2 - e^{-z} f. \tag{14}$$

There is no flow across the soil surface outside the channel so that on the soil surface on $z = 0$

$$\Psi - \frac{\partial \Psi}{\partial z} = 0. \tag{15}$$

The boundary condition over the surface of the channel with boundary with normal $\mathbf{n} = (n_1, n_2)$ is that of specified normal flow over the boundary surface. That is

$$-\left[\frac{\partial \Psi}{\partial x}n_1 - (\Psi - \frac{\partial \Psi}{\partial z})n_2\right] = \exp(-z)f_0(x,z) \quad \text{for} \quad (x,z) \in \partial\Omega_1, \tag{16}$$

where $\partial\Omega_1$ denotes the boundary of the channel and $f_0(x,z)$ is given. The boundary condition along the impermeable inclusion is that there is zero flux normal to the boundary.

4 Boundary integral equation

The boundary integral equation for the solution to equation (11) is given by (see Aziz, Clements and Lobo [1])

$$\lambda\Psi(a,b) = -\int_{\partial\Omega}\left[\frac{\partial\Psi}{\partial n}\phi' - \frac{\partial\phi'}{\partial n}\Psi\right]dS, \tag{17}$$

where $\mathbf{n}=(n_1,n_2)$ is the outward pointing normal to the domain Ω, $\lambda=1$ if $(a,b)\in\Omega$ and $\lambda=1/2$ if $(a,b)\in\partial\Omega$ (the boundary of Ω) and $\partial\Omega$ has a continuously turning tangent. In the case of equation (11) the ϕ' in equation (17) is given by

$$\phi'(x,z;a,b) = -\frac{1}{2\pi}K_0(r). \tag{18}$$

where $r=((x-a)^2+(z-b)^2)^{1/2}$ and K_0 is a modified Bessel function. Substitution of (14) into (17) gives

$$\lambda\Psi(a,b) = -\int_{\partial\Omega}\left[\phi' n_2 - \frac{\partial\phi'}{\partial n}\right]\Psi dS + \int_{\partial\Omega} f e^{-z}\phi'\,dS. \tag{19}$$

If the flux is zero across large sections of the soil surface on $z=0$ then in place of the fundamental solution (18) it is convenient to use Green's function given by Basha [2]

$$\begin{aligned}\phi'(x,z;a,b) &= -\frac{1}{2\pi}(K_0(r)+K_0(\bar{r})) \\ &\quad +\frac{1}{\pi}e^z\int_z^\infty e^{-\mu}K_0(\left[(x-a)^2+(z+\mu)^2\right]^{\frac{1}{2}})d\mu, \end{aligned}\tag{20}$$

where $\bar{r}=((x-a)^2+(z+b)^2)^{1/2}$. Using the transformation $z-\mu=-\eta$ equation (20) becomes

$$\begin{aligned}\phi'(x,z;a,b) &= -\frac{1}{2\pi}(K_0(r)+K_0(\bar{r})) \\ &\quad +\frac{1}{\pi}e^z\int_0^\infty e^{-\eta}K_0(\left[(x-a)^2+(z+\eta+b)^2\right]^{\frac{1}{2}})d\eta. \end{aligned}\tag{21}$$

With this choice of Green's function $\phi'-\partial\phi'/\partial z=0$ on $z=0$ so that equation (19) reduces to

$$\lambda\Psi(a,b) = -\int_{\partial\Omega_1}\left[\phi' n_2 - \frac{\partial\phi'}{\partial n}\right]\Psi dS + \int_{\partial\Omega_1} f e^{-z}\phi'\,dS. \tag{22}$$

An alternative boundary integral equation formulation which directly relates the potential θ and the flux f may be obtained as follows. From equation (10)

$$\Psi(x,z) = e^{-z}\theta(x,z). \tag{23}$$

Now let $\bar{\phi} = e^{b-z}\phi'$ then

$$\frac{\partial \bar{\phi}}{\partial n} = \frac{\partial (e^{b-z}\phi')}{\partial n} = e^{b-z}\left(\frac{\partial \phi'}{\partial n} - \phi' n_2\right). \tag{24}$$

Subtitution of equations (23) and (24) into (19) gives

$$\lambda\theta = \int_{d\Omega}\left(\frac{\partial \bar{\phi}}{\partial n}\theta + f\bar{\phi}\right) ds. \tag{25}$$

Now $\partial\bar{\phi}/\partial n = 0$ on $z = 0$ and $f = 0$ on the surface of the inclusion so (25) may be written in the form

$$\lambda\theta = \int_{\partial\Omega_1}\left(\frac{\partial \bar{\phi}}{\partial n}\theta + f\bar{\phi}\right) ds + \int_D \frac{\partial \bar{\phi}}{\partial n}\theta ds. \tag{26}$$

where D denotes the boundary of the impermeable inclusion.

5 Results and Discussion

In this section some numerical values of the matric flux potential θ associated with infiltration from one or two semicircular channels with inclusions configured as shown in Figures 1 and 2 are presented. The normal flux over the surface of the channel is chosen to be constant $F = -V_0$. Hence when the reference length L is chosen to be 100 cm and the empirical constant $\alpha = 0.002$ cm^{-1} (see Philip [6]) then from equation (6) the dimensionless value of the normal flux over the channel surface is

$$f = \frac{2}{V_0 \alpha L} F = -\frac{2}{0.2} = -10 \tag{27}$$

and the dimensionless value of the radius of the semicircular channel is

$$r = \frac{\alpha L}{\pi} = \frac{0.2}{\pi} = 0.0636 \tag{28}$$

The value of β is chosen to be 2.5 so that the dimensionless value of the depth z where the inclusion is placed is

$$z = \frac{\alpha Z}{2} = \frac{\alpha \beta L}{2} = 0.25 \tag{29}$$

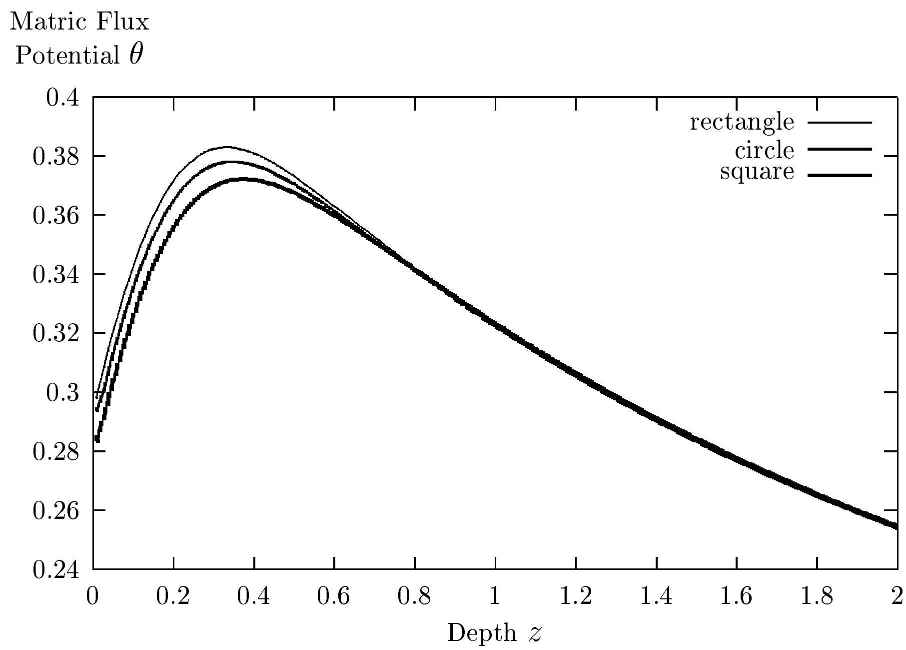

Figure 3: The value of the matric flux potential θ at $x = 0.5$ for a single semicircular channel with impermeable inclusions.

The dimensionless value of the area of each of the inclusions is identical and is taken to be $\pi/100$. For the single channel case the inclusion is centered at $(x, z) = (0.0, 0.25)$ with the radius of the circle 0.1 and the sides of the square 0.1771 while the rectangle is 0.2 in length and 0.1570 in width. For the double channel case the distance between the center of the inclusions is equal to the distance between the center of the channels. The narrower rectangle is twice the length and half the width of the original rectangle.

The boundary integral equation in (26) is used to compute the dimensionless values of θ along the line $x = 0.5$. The surface boundary outside the channel as well as the channel surface and the boundary of the impermeable inclusions were divided into segments in order to transform the integral in equation (26) to a system of linear algebraic equations for the unknown function $\theta(a, b)$. To obtain convergence of the dimensionless value of $\theta(a, b)$ to four decimal places the total number of segments for the one channel case was 270 and 350 for the double channel case.

Figure 3 illustrates the dimensionless value of the matric flux potential θ as a function of dimensionless depth z for a single semicircular channel with one inclusion. The results indicate that at $x = 0.5$ the value of θ is not significantly affected by the different shapes of the impermeable inclusion of the same area. However the rectangular inclusion produces slightly higher

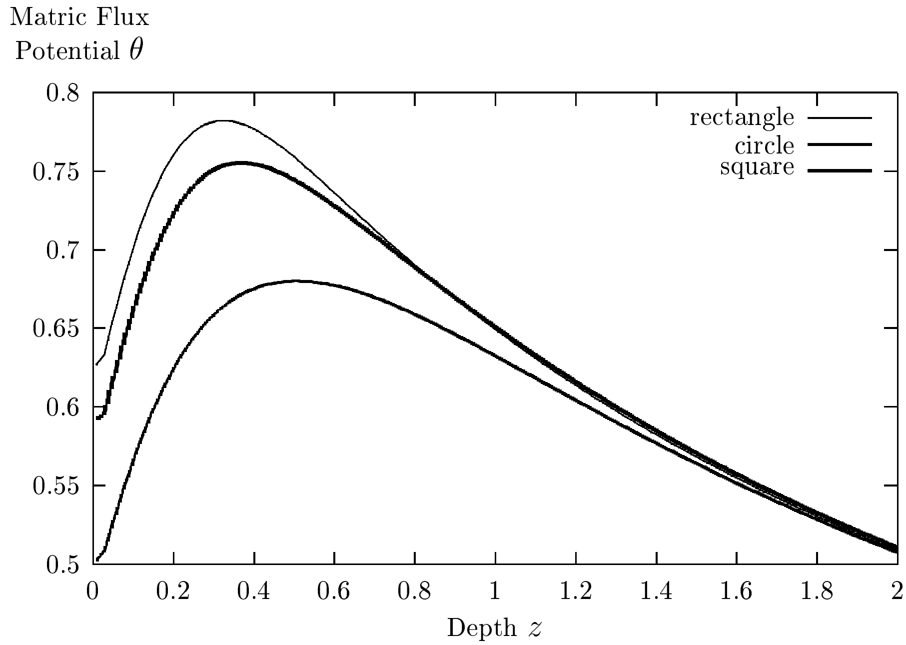

Figure 4: The value of the matric flux potential θ at $x = 0.5$ for double semicircular channels with various shapes of impermeable inclusions.

values of θ compared to the other shapes particularly in the region between the planes $z = 0.3$ and $z = 0.4$.

Figure 4 presents graphs of dimensionless values of θ for two semicircular channels with various inclusions. From the graphs we can observe that the value of the matric flux potential θ is approximately twice the value of θ for the single channel. This is to be expected since the flux from both channels has contributed to the increase in the value of θ at $x = 0.5$. The results also indicate that the rectangular inclusion produces the highest value of $\theta(\theta = 0.7822$ at $z = 0.33)$ followed by the square inclusion ($\theta = 0.7553$ at $z = 0.37$) and circular inclusion ($\theta = 0.6802$ at $z = 0.50$).

Figure 5 describes the value of θ from single and double semicircular channels with rectangular inclusions of the same area but different widths and lengths. The results indicate that how increasing the length of the rectangular inclusion increases the value of the matric flux potential along the line $x = 0.5$.

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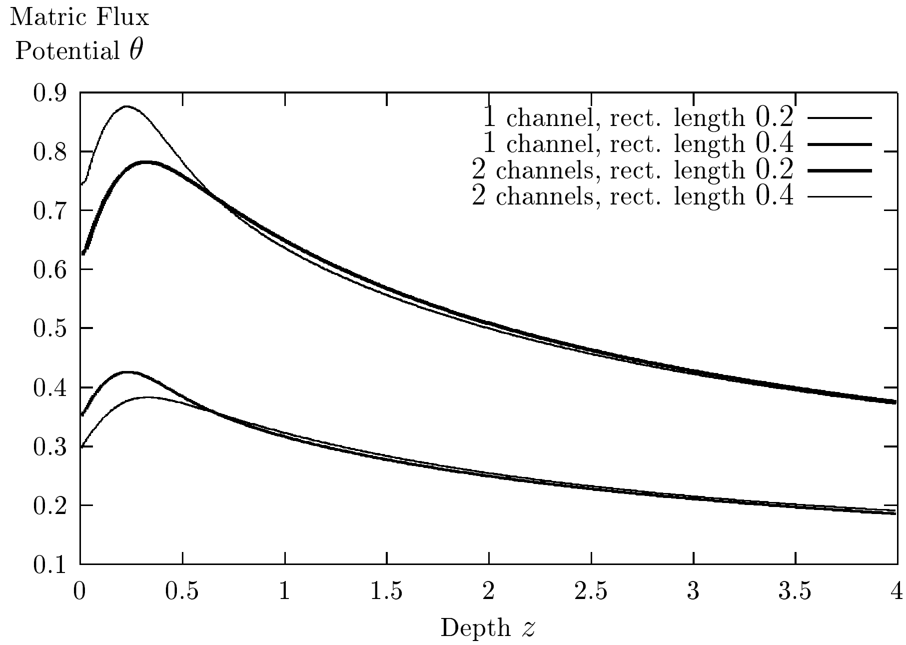

Figure 5: The value of the matric flux potential θ at $x = 0.5$ for single and double semicircular channels with rectangular impermeable inclusion

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