Numerical technique for linear and nonlinear eigenvalue problems in the theory of elasticity

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6 September 2004

Abstract

The numerical technique is based on a variational principle and iterative schemes. The algorithms solve a series of important eigenvalue problems for linear and nonlinear partial differential equations of bending of elastic plates. First we study a stress-strain state of the plate under Dirichlet conditions, using Galerkin projections. Then eigenfunctions of nonlinear equations, describing postbuckling behaviour of von Karman plate are designed. The effective computational technique allows us to detect bifurcation points and trace branches of the solutions. The plate is supposed to be simply supported, clamped and compressed along its four sides. The basis functions in the variational-spectral approach are Legendre polynomials and trigonometrical functions. They are chosen in correspondence with the boundary conditions. Some numerical examples demonstrate efficiency of the methods. The proposed algorithms and obtained results are applicable to similar problems and elliptic type differential equations.

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1 Introduction

The aim of this paper is to demonstrate a numerical robust technique for solving of a class of problems, describing a behaviour of thin elastic rectangular plates. The basic idea lies in variational-projective studies of the mechanical models. We apply Galerkin spectral method for finding of eigenfunctions.

In the second section we consider Dirichlet problem about compression and tension of the plate. The model is treated by the variational discrete method with Legendre polynomials. The divided differences with respect to the indexes of Legendre polynomials, used in this paper, in the integral form for ordinary differential equations were first introduced by Mikhlin [8]. Later this approach was developed for some multi-dimensional boundary value problems by Vashakmadze [10].

The next part of the paper is devoted to a numerical analysis of von Karman problem in buckling. We consider a simply supported, partially and totally clamped plate, subjected to a uniform lateral compression on its four sides. The bifurcation phenomenon is analyzed for these cases.

Postbuckling behaviour of von Karman plate was investigated by several authors. There are many numerical approaches, treating the mechanical models. Basically, they are finite elements and difference methods. The technique for discrete schemes is based on the Newton, GMRES algorithms (e.g. [2, 5]) and sophisticated numerical continuation ([1]). For instance, Chien, Chang, Mei [2] have applied GMRES algorithm in context of numerical continuation for the plate, compressed on its two sides. The most general case of bifurcation phenomenon was studied numerically by Allgower, Georg [1]. In the works of Holder, Schaeffer [7] and Schaeffer, Golubitsky [9] one has shown that so-called mode jumping, when the primary solution branches lose stability through further bifurcation "may occur under the partially but not for the simply supported conditions". Obviously, mode jumping can also occur under totally clamped ones. In recent paper of Dossou, Pierre [5] deformation and bifurcation for discrete von Karman problem for the totally clamped plate is excellent analyzed by Newton-GMRES approach.

In the present paper we propose a spectral method with a choice of global trial functions span the whole domain. This approach is different from traditional finite elements. It easily allows to estimate error of approximations, guarantees a high accuracy and computational efficiency.

2 Linear model of stress-strain state of the plate

The eigenvalue Dirichlet problem for linear partial differential equations of compression and tension of an elastic plate has the following form:

$$\begin{gathered} -k_1 \Delta u(x,y) - k_2 \operatorname{grad}\operatorname{div} u(x,y) = \lambda u(x,y) \quad (x,y) \in G\,, \\ u(x,y) = 0 \quad (x,y) \in \partial G\,, \end{gathered} \tag{1}$$

where $u = (u_1, u_2)^T$ is a compression and tension function; k_1, k_2 are positive constant coefficients, depending on the rigidity of the plate; G is an open subset in R^2, occupied by the plate.

According to the variational method the solution of (1) is found as

$$\overset{n}{u} = \sum_{i,j=1}^{N} \overset{n}{u}{}^{ij} \varphi_{ij}(x,y)\,, \tag{2}$$

where φ_{ij} are coordinate(trial) functions.

Let $G = [-1,1] \times [-1,1]$ (by changing of the variables $x = \frac{l_1}{2}(x_1+1)$, $y = \frac{l_2}{2}(y_1+1)$ we can lead (1) on $G = [0,l_1] \times [0,l_2]$ to the problem on the square $[-1,1]^2$). Then φ_{ij} in (2) is chosen as the first order divided differences with respect to the indexes of orthogonal Legendre polynomials $\chi P_i(x) = \frac{1}{\sqrt{2(2i+1)}}(P_{i+1} - P_{i-1})$, which is a complete and closed system in Sobolev space $W_2^1(G)$ and obtained from the integral form of the normalized Legendre polynomials ([8]) $\Phi_i(x) = \sqrt{\frac{2i+1}{2}} \int_{-1}^{x} P_i(t)dt \quad (i = 1, 2, \dots)$ by use the property

$$P_i(x) = \frac{1}{2i+2}(P'_{i+1}(x) - P'_{i-1}(x))\,. \tag{3}$$

After applying Galerkin projections we have a system of linear algebraic equations with respect to $\overset{n}{u}{}^{ij}$,

$$A\overset{n}{u} = \lambda B\overset{n}{u}\,. \tag{4}$$

Here $A^{ij}\overset{n}{u} = (L\overset{n}{u}, \psi_{ij}(x,y))$, $(A = (A_1, A_2)^T)$, $B^{ij}\overset{n}{u} = (\overset{n}{u}, \psi_{ij}(x,y))$, $(B = B_1, B_2)^T)$ $(\{\psi_{ij}\}$ is a complete in $L_2(G)$ enlarging φ_{ij} by [10]).

Table 1: u_1^{ij}, $i, j = 1, 3, \ldots, 9$.

i,j	1	3	5	7	9
1	2.40853	-0.36630	0.02007	0.00116	0.00042
3	-0.31270	0.22711	0.04542	0.01386	0.00399
5	0.01219	0.03754	0.05688	0.03100	0.01123
7	0.00027	0.00666	0.02306	0.02564	0.01449
9	0.00004	0.00101	0.00586	0.01156	0.01017

Table 2: u_1^{ij}, $i, j = 2, 4, \ldots, 8$.

i,j	2	4	6	8
2	0.90657	-0.03058	0.00568	0.00085
4	0.09761	0.10490	0.02714	0.00561
6	0.01886	0.05357	0.03847	0.01380
8	0.00412	0.01679	0.02187	0.01259

Eigenfunctions of Laplacian under Dirichlet conditions are also used as the basis, exactly $\varphi_{ij}(x,y) = \frac{2}{\sqrt{l_1 l_2}} \sin(\frac{\pi i x}{l_1}) \sin(\frac{\pi j y}{l_2})$, $x = [0, l_1]$, $y = [0, l_2]$. Then we obtain pertaining analogue formulae.

(4) is split into four independent subsystems, subject from eveness of indexes (i,j). A_l and B_l $(l = 1,2)$ are positive definite band symmetric matrices. Cholesky decomposition is applied to B and therefore $B = LL^T$. Then

$$L^{-1}A(L^{-1})^T \overset{n}{v} = \lambda \overset{n}{v}, \quad \overset{n}{v} = L^T \overset{n}{u}, \quad \overset{n}{u} = (L^T)^{-1} \overset{n}{v}. \tag{5}$$

In present time there are many software (LinearAlgebra libraries in Python codes, Maple, Mathematica and etc.) able quickly compute eigenvalues and eigenfunctions of (5). Below we illustrate a numerical example.

Example. $k_1 = 1$, $k_2 = 1$ with $N = 9 : \lambda_1^{11} = 7.231$ (it is sufficient $N = 9$ to get error of approximation less than 10^{-5} for the first 9th eigenvalues λ_1^{ij}, $i, j = 1, 2, 3$). The components of $u(x, y; \lambda_1^{11})$ are given in Tables 1, 2.

3 Eigenvalue problem for von Karman plate

3.1 Mechanical formulation

In this section we consider nonlinear equations of bending and stretching of a rectangular plate with buckling. The two-dimensional model reads:

$$\begin{array}{rcl} D\Delta^2 w &=& 2h_0[w,\psi] + \lambda[\theta, w] \quad (x,y) \in G\,, \\ \Delta^2 \psi &=& -\frac{E}{2}[w,w]\,, \end{array} \tag{6}$$

where $[w,\psi] = \dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 \psi}{\partial y^2} + \dfrac{\partial^2 \psi}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} - 2\dfrac{\partial^2 w}{\partial x \partial y}\dfrac{\partial^2 \psi}{\partial x \partial y}$, $w(x,y)$ denotes the deflection, $\psi(x,y)$ is the Airy stress potential, the parameter θ is a regular function defined on G, which values depend on the portion of the boundary, subjected to compression on the boundary conditions, λ is the intensity of the compression, $D = \frac{2Eh_0^2}{3(1-\nu^2)}$ is a cylindrical rigidity, E is Joung modulus, h_0 is a thickness and ν is Poisson ratio. If θ is compression on all ends

$$\theta(x,y) = -\frac{1}{2}(x^2+y^2)\,. \tag{7}$$

Compression only on two sides implies

$$\theta(x,y) = -\frac{1}{2}x^2 \quad \text{or} \quad \theta(x,y) = -\frac{1}{2}y^2\,. \tag{8}$$

Supposing all physical parameters in (6) equal unit (the technique is applicable for different values of D, h_0, E), which is valid for polygonal plates.

The plate, subjected to a compressed on its two ends (8_2) was regarded in [7, 9, 2, 3] and others. We shall study the case, when the plate is compressed on its four sides, i.e., (7) holds. Thus, (6) is rewritten as

$$\begin{aligned} \Delta^2 w &= [w,\psi] - \lambda \Delta w \quad (x,y) \in G\,, \\ \Delta^2 \psi &= -[w,w]\,. \end{aligned} \tag{9}$$

Let $L_1 w = 0\,, \quad L_2\psi = 0$ be boundary conditions on the edges. According to the results of [4] if $\lambda \le \lambda_1$, where λ_1 is the first eigenvalue of the linearized(spectral) problem

$$\begin{gathered} \Delta^2 w + \lambda \Delta w = 0 \quad (x,y) \in G\,, \\ L_1 w = 0 \quad (x,y) \in \partial G\,, \end{gathered} \tag{10}$$

then (9) has a unique trivial solution. If $\lambda > \lambda_1$ then (9) has at least three solutions (w,ψ), $(-w,\psi)$, $(w \neq 0,\ \psi \neq 0)$ and $(0,0)$.

When the plate is simply supported the classical boundary conditions are

$$\begin{aligned} (a) \quad w &= 0\,, \quad \Delta w = 0 \quad (x,y) \in \partial G\,, \\ (b) \quad \psi &= 0\,, \quad \Delta \psi = 0 \quad (x,y) \in \partial G \end{aligned} \tag{11}$$

or more appropriate condition in physical sense for the stress function ([9])

$$\frac{\partial \psi}{\partial n} = \frac{\partial}{\partial n}\Delta\psi = 0 \quad (x,y) \in \partial G\,. \tag{12}$$

If the plate is partially clamped then

$$w = \begin{cases} \frac{\partial w}{\partial n} = 0\,, & x = 0, l_1\,, \\ \Delta w = 0\,, & y = 0, l_2\,. \end{cases} \tag{13}$$

The conditions (13), (11b) and (13), (12) correspond to the clamped sides $x = 0, l_1$ and simply supported ends $y = 0, l_2$.

For the totally clamped one:

$$\begin{aligned} w &= \frac{\partial w}{\partial n} = 0 \quad (x,y) \in \partial G\,, \\ \psi &= \frac{\partial \psi}{\partial n} = 0 \quad (x,y) \in \partial G\,. \end{aligned} \tag{14}$$

3.2 Eigenvalues of the linearized problem

The first step of numerical analysis of (9) is to detect primary(main) bifurcation points. Actually, we need to compute eigenvalues of the linearized problem (10), which will be points where solution of (9) bifurcates from trivial one. The corresponding branches we call primary branches and which might bifurcate from them are called secondary ones. For simply supported plate (11) the linearized problem (10) has nontrivial solutions when λ are eigenvalues of Laplacian under Dirichlet conditions, i.e., $\lambda_{mn} = \pi^2\big(\big(\frac{m}{l_1}\big)^2 + \big(\frac{n}{l_2}\big)^2\big)$ and corresponding eigenfunctions $U_{mn}(x,y) = \sin\frac{m\pi x}{l_1}\sin\frac{n\pi y}{l_2}$. No such formulae exist for the other boundary conditions. To find eigenvalues of (10) for partially clamped plate a rule of separation of variables is applied, i.e., $\omega(x,y) = u(x)v(x) \neq 0$. Substituting the last expression into (10) one gets

$$u''''v + 2u''v'' + uv'''' + \lambda(u''v + uv'') = 0 \tag{15}$$

with boundary conditions

$$u(0) = u(l_1) = 0\,, \quad u'(0) = u'(l_1) = 0\,, \tag{16}$$

$$v(0) = v(l_2) = 0\,, v''(0) = v''(l_2) = 0\,. \tag{17}$$

Choosing the basis $\sin\frac{\pi n}{l_2}$: $n \in N$ for the space of functions $v \in C^4[0, l_2]$ with (17) we reduce (15) to an ordinary differential equation. The general solution of this equation is $u(x) = c_1 e^{\alpha_1 x} + c_2 e^{-\alpha_1 x} + c_3\cos\alpha_2 x + c_4 \sin\alpha_2 x$, where $\alpha_1 = \pi n/l_2$, $\alpha_2 = \sqrt{\lambda - (\pi n/l_2)^2}\,(\lambda > (\pi n/l_2)^2)$. Using the boundary conditions (16) we obtain a system with respect to c_1, c_2, c_3, c_4. It has a nontrivial solution if the determinant of the coefficients equals zero, i.e.,

$$\begin{aligned} &2\alpha_2 + e^{\alpha_1 l_1}\left[(\alpha_1 - \lambda/2\alpha_1)\sin\alpha_2 l_1 - \alpha_2\cos\alpha_2 l_1\right] \\ &-e^{-\alpha_1 l_1}\left[(\alpha_1 - \lambda/2\alpha_1)\sin\alpha_2 l_1 + \alpha_2\cos\alpha_2 l_1\right] = 0\,. \end{aligned}$$

Solving the last equation with respect to λ we define eigenvalues of (10).

For the totally clamped plate Galerkin procedure is applied. The solution is found as partial sums of double series

$$W_N(x,y) = \sum_{i,j=1}^{N} w_N^{ij}\omega ij(x,y)\,. \tag{18}$$

The second order divided differences with respect to the indexes of Legendre polynomials $\chi^2 P_i(x) = \sqrt{(2i+1)/2}(a_i P_{i+2}(x) + c_i P_i(x) + b_i P_{i-2}(x)$, where $a_i = \frac{1}{(2i+1)(2i+3)}$, $c_i = -\frac{2}{(2i-1)(2i+3)}$, $b_i = \frac{1}{(2i-1)(2i+1)}$ are taken as the trial functions $\omega_{ij}(x,y)$ assuming $G = [-1,1]^2$. They are derived by applying twice the property (3) (see Section 2) in the integral relation $\Phi_i(x) = \sqrt{\frac{2i+1}{2}} \int_{-1}^{x} dt \int_{-1}^{t} P_i(\tau) d\tau$ ([8]).

For our purposes, mainly for the nonlinear equations, it is more convenient to use combinations of trigonometrical functions. Therefore, we introduce a new basis for (10), which will be applied for the nonlinear problem as well. The basis is $\omega_{ij}(x,y) = \chi\phi_i(x,l_1)\chi\phi_j(y,l_2)$, where $\chi\phi_i(x,l) = \sqrt{\frac{2}{l}}(\phi_{i+1}(x,l) - \phi_{i-1}(x,l))$, $\phi_i(x,l) = \cos\frac{\pi i x}{l}$, $i = 0,1,\ldots$, $\frac{2}{l}\|\phi_i\|^2 = \frac{2}{l}\int_0^l \phi_i^2(x,l)dx = 1$ and $\phi_i^{(v)}(x,l) = \left(\frac{\pi i}{l}\right)^v \phi_i(x,l)$. $\{\chi\phi_i(x,l)\}$ is a linear independent, complete in Sobolev space and satisfy the boundary conditions (14)[1].

By the Galerkin method we get a system of algebraic equations

$$K_N^{mn} w_N = \lambda B_N^{mn} w_N, \tag{19}$$

where K_N and B_N are discretizations of the biharmonic and harmonic operators correspondingly. (19) is split into four subsystems, each of them might be solved separately. The algorithm has been implemented in `Numerical Python` codes with use of `LinearAlgebra.py` module of Python's library and built-in `Cholesky_decomposition`. Under $l_1 = l_2 = 1$, $N = 10$ the computed eigenvalues λ_{mn} $(m,n = 1,2,\ldots,6)$ are:

52.358	92.157	92.157	128.302	154.226	167.066	189.790	189.790
246.416	246.416	246.821	269.503	279.298	327.380	327.380	349.570
362.827	380.825	380.825	404.956	426.889	435.772	481.918	481.918
501.837	505.848	505.848	512.328	556.020	556.020	604.095	624.334
625.188	632.396	638.823	654.424				

3.3 The spectral method for the nonlinear model

The technique, introduced in Subsection 3.2, we extend for (9). That means

$$W_N(x,y) = \sum_{i,j=1}^{N} w_N^{ij}\omega_{ij}(x,y), \quad \Psi_N(x,y) = \sum_{i,j=1}^{N} \psi_N^{ij}\varphi_{ij}(x,y). \tag{20}$$

Here for simply supported conditions (11a), (11b) or (12) $\omega_{ij}(x,y) = \varphi_{ij}(x,y) = \frac{2}{\sqrt{l_1 l_2}} \sin\frac{\pi i x}{l_1} \sin\frac{\pi j y}{l_2}$ or $\varphi_{ij}(x,y) = \frac{2}{\sqrt{l_1 l_2}} \cos\frac{\pi i x}{l_1} \cos\frac{\pi j y}{l_2}$. If the plate is partially

[1] Note, that $\phi_i(x,l) - \phi_{i+2}(x,l)$ are eigenfunctions of an ordinary differential equation, coming up after the rule of separation to $\Delta^2 w + \lambda w_{xx} = 0$ (e.g. [7, 9, 2]).

clamped (13): $\omega_{ij}(x,y) = \chi\phi_i(x,l_1)\sqrt{\frac{2}{l_2}}\sin\frac{\pi j y}{l_2}$ and $\varphi_{ij}(x,y)$ is determined as before. When we have the totally clamped case (14): $\omega_{ij}(x,y) = \varphi_{ij}(x,y) = \chi\phi_i(x,l_1)\chi\phi_j(y,l_2)$. After applying the Galerkin projective method to (9) we derive nonlinear algebraic equations

$$\begin{aligned} K^{mn}_{1,N} w_N &= A^{mn}_{1,N}(w_N, \psi_N) + \lambda B^{mn}_N w_N, \\ K^{mn}_{2,N} \psi_N &= -A^{mn}_{2,N}(w_N, w_N). \end{aligned} \tag{21}$$

Here $K_{1,N}$, $K_{2,N}$, B_N are linear and $A_{1,N}$, $A_{2,N}$ nonlinear discrete matrices.

3.4 The iterative scheme and numerical continuation

We solve (21) by Newton and numerical continuation algorithms with incremental loading parameter λ. Note, l_1, l_2 might also be loading parameters.

The secondary bifurcation points can be studied by Newton-based method with rationally computed quantities, suggested by Griewank, Osborne in [6]. In [5] is also described the algorithm of detecting singular points along solution curves. The secondary bifurcation often results from a double eigenvalue.

Let $\lambda \in [\lambda_0, \lambda_0 + \Lambda]$ ($\lambda_0 = \lambda_{11} + \varepsilon$. Divide $[\lambda_0, \lambda_0 + \Lambda]$ by M parts so that $\lambda_r = \lambda_{r-1} + \delta_r$, $r = 1, 2, \ldots, M$, $\lambda_M = \lambda_0 + \Lambda$. To start Newton iterations for (21) we choose $\lambda = \lambda_0$, further, use the results for next λ_1 and continue so on. Below we demonstrate an effective algorithm, thanks to which the primary branches of the eigenfunctions are traced. Let η_i, $i = 1, 2, 3, \ldots$ be eigenvalues of (10) with $\eta_i < \eta_{i+1}$ then the algorithm in stages looks as:

I. $\eta_1 < \lambda \le \eta_2$:

1. $w_N^{i_1 j_1} = c(l_1, l_2)\,, \quad w_N^{ij} = 0\,, \quad i \ne i_1\,, \quad j \ne j_1$

II. $\eta_2 < \lambda \le \eta_3$:

1. $w_N^{i_1 j_1} = \overset{r_0\gamma_0}{w_N^{i_1 j_1}}$ ($\overset{r_0\gamma_0}{w_N^{i_1 j_1}}$ is computed on the step I,1)
2. $w_N^{i_2 j_2} = c(l_1, l_2)\,, \quad w_N^{ij} = 0\,, \quad i \ne i_2\,, \quad j \ne j_2$

III. $\eta_3 < \lambda \le \eta_4$:

1. $w_N^{i_1 j_1} = \overset{r_1\gamma_1}{w_N^{i_1 j_1}}$ ($\overset{r_1\gamma_1}{w_N^{i_1 j_1}}$ is computed on the step II,1)
2. $w_N^{i_2 j_2} = \overset{r_0\gamma_0}{w_N^{i_2 j_2}}$ ($\overset{r_0\gamma_0}{w_N^{i_2 j_2}}$ is computed on the step II,2)
3. $w_N^{i_3 j_3} = c(l_1, l_2)\,, \quad w_N^{ij} = 0\,, \quad i \ne i_3\,, \quad j \ne j_3$

end etc.

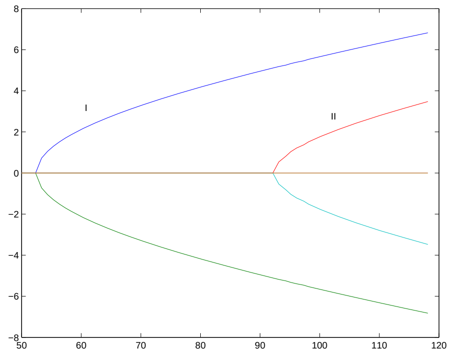

Figure 1: $l_1 = 1\,,\ l_2 = 1\,,\ N = 3\,,\ \eta_1 = \lambda_{11}\,,\ \eta_2 = \lambda_{12} = \lambda_{21}$.

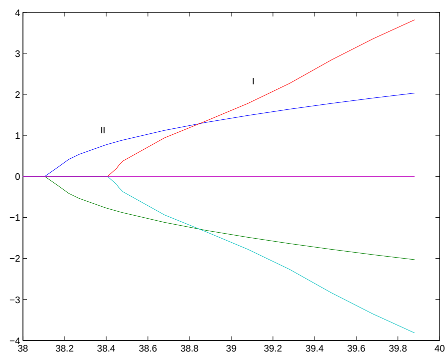

Figure 2: $l_1 = 3\,,\ l_2 = 1\,,\ N = 4\,,\ \eta_1 = \lambda_{21}\,,\ \eta_2 = \lambda_{11}$.

Here $w^{i_k j_k}$ are the coefficients in the dominating modes of the expansion (20). For example, for the simply supported plate with ($l_1 = l_2 = 1$) we can take value $i_1 = j_1 = 1$ because $U_{11}(x, y) = \sin \pi x \sin \pi y$ is the eigenfunction for $\eta_1 = \lambda_{11} = 2\pi^2$ and put $c(l_1, l_2) \equiv 1$.

Figures 1, 2 show the maximum of deflection of the first two branches and their symmetrical ones of the solution for the totally clamped plate with different sizes. The maximum of deflection implies $\max\limits_{G_M} W_N(x_i, y_j)$ or $\min\limits_{G_M} W_N(x_i, y_j)$, where $G_M = \{(x_i, y_i)\,,\ i, j = 1, 2, \ldots, M\}$ is a mesh on G.

Acknowledgements: The author is grateful to Prof. Mike Osborne (ANU) for discussions and useful advices and Assoc. Prof. George Stavroulakis (UOI), Prof. Tamaz Vashakmadze (TSU) for the interest to this work.

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