# A New Adaptive GMRES($m$) Algorithm with Correction

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**Abstract**

GMRES($m$) method is a Krylov subspace method for solving nonsymmetric linear systems of equations. The difficulty of this method lies in choosing the appropriate restart cycle $m$. In this paper, we propose a new strategy of adaptive restart for GMRES($m$) method which based on an idea due to the difference of the Ritz and harmonic Ritz values. We also report on numerical experiments. These results show that this new approach is both effective and robust.

## Contents



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# 1 Introduction

In this paper, we address the problem of solving a linear system of the form

$$A\boldsymbol{x} = \boldsymbol{b}, \tag{1}$$

where $A \in \mathcal{C}^{n\times n}$ is a sparse nonsymmetric and nonsingular matrix, and $\mathcal{C}^{n\times n}$ denotes the set of complex matrices of dimension $n$. Such linear systems often come up in lots of areas of scientific computing.

GMRES method is one of the Krylov subspace method for solving linear system (1). It minimizes the residual norm over the Krylov subspace

$$\mathcal{K}_m(A, \boldsymbol{r}_0) = \mathrm{SPAN}\{\boldsymbol{r}_0, A\boldsymbol{r}_0, A^2\boldsymbol{r}_0, \ldots, A^{(m-1)}\boldsymbol{r}_0\}, \quad m = 1, 2, \ldots \tag{2}$$

at every step. $\boldsymbol{r}_0$ is the initial residual vector. In GMRES method, the orthonormal basis $(v_1, v_2, \ldots, v_m)$ of $\mathcal{K}_m(A, \boldsymbol{r}_0)$ is computed by orthogonalizing the Krylov basis $\{\boldsymbol{r}_0, A\boldsymbol{r}_0, \ldots, A^{(m-1)}\boldsymbol{r}_0\}$ with Arnoldi process [1]. The orthonormal basis vectors $V_m = (v_1\ v_2\ \ldots\ v_m) \in \mathcal{C}^{n\times m}$ form an orthogonal matrix. In the Arnoldi process, scalars $h_{i,j}$ are also computed so that the square upper Hessenberg matrix $H_m = (h_{i,j}) \in \mathcal{C}^{m\times m}$ satisfies

$$AV_m = V_m H_m + h_{m+1,m} v_{m+1} \boldsymbol{e}_m^H = V_{m+1}\bar{H}_m, \tag{3}$$

where $\bar{H}_m \in \mathcal{C}^{(m+1)\times m}$ is the matrix $H_m$ supplemented with an extra row $(0 \ldots 0\ h_{m+1,m})$. $\boldsymbol{e}_m$ is column $m$ of the identity matrix. Multiplying (3) by $V_m^H$ from the left, we get

$$V_m^H A V_m = H_m. \tag{4}$$

The upper Hessenberg matrices $H_m$ and $\bar{H}_m$ can be used for computing the Ritz and harmonic Ritz values.

In theory, GMRES method converges before $n$ iterations. The amount of work required for orthogonalizing and the memory required for saving the vectors $(v_1, v_2, \ldots, v_n)$ and the upper Hessenberg matrix $\bar{H}_m$ in per iteration increase linearly with the iteration count. According to this drawback, GMRES method is impractically for large linear system. In order to reduce the cost required by full GMRES method, restarted version of GMRES method, denoted by GMRES($m$), is often used. The algorithm restarts after each cycle of $m$ iterations. Comparing to the full GMRES method, GMRES($m$) method requires a little work and storage, but the difficulty lies in choosing the appropriate restart cycle $m$. Generally, $m$ is selected according to numerical experience. The best way for the selection of $m$ hasn't been proposed yet. A number of different approaches of adaptive restarting have been proposed by the several authors [6, 11, 12, 15]. In this paper, we will propose a new possible adaptive restart strategy which based on an idea due to the difference of the Ritz and harmonic Ritz values which come from the upper Hessenberg matrices $H_m$ and $\bar{H}_m$.

In section 2, we briefly describe the Ritz and harmonic Ritz values and how to compute them from Krylov subspace. In section 3, a new restart strategy based on exploiting the idea of making use of the difference of the Ritz and harmonic Ritz values is proposed. This restart strategy can still work well in preconditioned GMRES($m$) method. As an example, we make use of the preconditioner used in DEFLATED-GMRES($m$, $k$) method in section 4. We also report on some numerical experiments comparing the resulting proposed algorithm with other algorithms in section 5. These results show that this new approach is both effective and robust.

## 2 Ritz and harmonic Ritz values

The Ritz and harmonic Ritz values and vectors are both approximate eigenvalues and eigenvectors. Recently, They have been used in a number of modified versions of GMRES method. For example, the harmonic Ritz values and harmonic Ritz vectors are used in MORGAN($m$, $k$) method [9, 10]. In this paper, we make use of the difference of the Ritz and harmonic Ritz values.

**Definition 2.1 (Ritz values [14]):** *If $\mathcal{V}_k$ is a linear subspace of $\mathcal{C}^n$, then $\lambda$ is a Ritz value of $A$ with Ritz vector $\boldsymbol{u}_k$ if*

$$\boldsymbol{u}_k \in \mathcal{V}_k, \quad \boldsymbol{u}_k \neq 0, \quad (A\boldsymbol{u}_k - \lambda \boldsymbol{u}_k) \ \perp \ \mathcal{V}_k \tag{5}$$

In the context of GMRES method, $\mathcal{V}_k$ is the Krylov subspace $\mathcal{K}_m(A, \boldsymbol{r}_0)$. A Ritz value $\lambda$ with Ritz vector $\boldsymbol{u} = V_m \boldsymbol{y}_m$, $\boldsymbol{y}_m \in \mathcal{C}^{m\times 1}$, respect to $\mathcal{K}_m(A, \boldsymbol{r}_0)$ satisfies

$$(A\boldsymbol{u} - \lambda \boldsymbol{u}) \perp \mathcal{K}_m(A, \boldsymbol{r}_0) \Leftrightarrow V_m^H (AV_m \boldsymbol{y}_m - \lambda V_m \boldsymbol{y}_m) = 0,$$

where $V_m$ is the same one in equation (4). Using relation (4), we have

$$H_m \boldsymbol{y}_m = \lambda \boldsymbol{y}_m \tag{6}$$

That is, the eigenvalues of $H_m$ are the Ritz values respect to Krylov subspace $\mathcal{K}_m(A, \boldsymbol{r}_0)$.

**Definition 2.2 (Harmonic Ritz values [14]):** *A Value $\bar{\lambda}$ is a harmonic Ritz value of $A$ with respect to some linear subspace $\mathcal{W}_k$ if $\bar{\lambda}^{-1}$ is a Ritz value of $A^{-1}$ respect to $\mathcal{W}_k$.*

In order to compute the harmonic Ritz values from Krylov subspace $\mathcal{K}_m(A, \boldsymbol{r}_0)$, we make use of the following theorem.

**Theorem 2.1 (Sleijpen et al. [14]):** *Let $\mathcal{V}_k$ be some k-dimensional subspace with basis $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, ..., $\boldsymbol{v}_k$. A value $\bar{\lambda}_k \in \mathcal{C}$ is a harmonic Ritz value of $A$ with respect to the subspace $\mathcal{W}_k := A\mathcal{V}_k$ if and only if*

$$A\boldsymbol{u}_k - \bar{\lambda}\boldsymbol{u}_k \perp A\mathcal{V}_k \quad \textit{for some } \boldsymbol{u}_k \in \mathcal{V}_k, \quad \boldsymbol{u}_k \neq \boldsymbol{0} \tag{7}$$

According to this theorem, a harmonic Ritz value $\bar{\lambda}$ with harmonic Ritz vector $\boldsymbol{u} = V_m\boldsymbol{y}_m$, $\boldsymbol{y}_m \in \mathcal{C}^{m\times 1}$, respect to subspace $A\mathcal{K}_m(A, \boldsymbol{r}_0)$ satisfies

$$(A\boldsymbol{u} - \bar{\lambda}\boldsymbol{u}) \perp A\mathcal{K}_m(A, \boldsymbol{r}_0) \Leftrightarrow (AV_m)^H(AV_m\boldsymbol{y}_m - \bar{\lambda}V_m\boldsymbol{y}_m) = 0,$$

where $V_m$ is the same one in equation (4). Using equation (3) and (4), we have $\bar{H}_m^H\bar{H}_m\boldsymbol{y}_m = \bar{\lambda}H_m^H\boldsymbol{y}_m$. When $H_m$ is nonsingular, it is rewritten as

$$H_m^{-H}\bar{H}_m^H\bar{H}_m\boldsymbol{y}_m = \bar{\lambda}\boldsymbol{y}_m \tag{8}$$

That is, the harmonic Ritz values respect to subspace $A\mathcal{K}_m(A, \boldsymbol{r}_0)$ are the eigenvalues of $H_m^{-H}\bar{H}_m^H\bar{H}_m$. Equation (3) allows us to rewrite $H_m^{-H}\bar{H}_m^H\bar{H}_m$ as

$$H_m + h_{m+1,m}^2 f_m \boldsymbol{e}_m^H, \tag{9}$$

where $f_m = H_m^{-H}\boldsymbol{e}_m$.

# 3 An adaptive restarting scheme for GMRES($m$)

According to the already stated, we know (i) eigenvalues of matrix $H_m$ are the Ritz values respect to Krylov subspace $\mathcal{K}_m(A, \boldsymbol{r}_0)$ and (ii) eigenvalues of $H_m + h_{m+1,m}^2 f_m \boldsymbol{e}_m^H$ are the harmonic Ritz values respect to subspace $A\mathcal{K}_m(A, \boldsymbol{r}_0)$. When an invariant Krylov subspace has been found, the harmonic Ritz values equal to the Ritz values, since in this case $h_{m+1,m} = 0$. From Saad et al. [13], GMRES method converges, when $h_{m+1,m}$ equals to zero. If it is not the case, we now consider the following equation.

$$\begin{aligned} \| \boldsymbol{f}_m \|_2 &= \| H_m^{-H}\boldsymbol{e}_m \|_2 \\ &\leq \| H_m^{-H} \|_2 \| \boldsymbol{e}_m \|_2 = 1/\sigma_{min}(H_m) \end{aligned} \tag{10}$$

where $\sigma_{min}(H_m)$ is the smallest singular value of $H_m$. Therefore, the upper bound of 2-norm of the second item in (9) is

$$\| h_{m+1,m}^2 \boldsymbol{f}_m \boldsymbol{e}_m^H \|_2 \leq \frac{h_{m+1,m}^2}{\sigma_{min}(H_m)} \tag{11}$$

This equation (11) shows that the difference between the Ritz and harmonic Ritz values can only be large when $|h_{m+1,m}|$ is large or $\sigma_{min}(H_m)$ is small, which is the case when GMRES method stagnates [5]. Therefore, The difference between the Ritz and harmonic Ritz values can be used for evaluating the stagnation of GMRES method.

Since it is difficult to determine a value is large or not, we take notice of the change of the difference between the Ritz and harmonic Ritz values. When the difference becomes larger, that is the possibility of the stagnation of GMRES method becomes larger. In order to avoid the possibility of stagnation, we consider an adaptive restart strategy by using the change of the difference between the Ritz and harmonic Ritz values. For details, we compute the difference between the Ritz and harmonic Ritz values per iteration. If the difference is larger than the one generated from the last iteration, approximate resolution $\boldsymbol{x}_m$ is computed, and it is used as $\boldsymbol{x}_0$ for creating the new Krylov subspace. That is restarting the GMRES method. When it is not the case, no restart is carried out. Note that the maximum restart cycle should be specified in view of memory's limitation. Minimum restart cycle can be specified too.

Next, we consider a procedure for the numerical estimation of the difference of the Ritz and harmonic Ritz values. Now, we take the absolute value of the difference of the max Ritz value $\lambda_{\max}$ and the max harmonic Ritz value $\bar{\lambda}_{\max}$. In this paper, the new method is denoted as RITZ-GMRES($m_{\min}$, $m_{\max}$), where $m_{\min}$ is the minimum restart cycle and $m_{\max}$ is the maximum restart cycle. Algorithm of RITZ-GMRES($m_{\min}$, $m_{\max}$) method is given in Figure 1. Comparing with the classical GMRES($m$) method, extra work for computing $\lambda_{\max}$ and $\bar{\lambda}_{\max}$ is required in RITZ-GMRES($m_{\min}$, $m_{\max}$) method. But the cost only depends on the dimension of the Krylov subspace, which is really small ($\leq m_{\max}$).

In section 5, we show that this new algorithm can avoid unnecessary iterations by using the adaptive restarting scheme with example 1, 2.

# 4 Application in preconditioned GMRES($m$) method

The new restart strategy proposed in the above section can also work well in preconditioned GMRES($m$) methods. For example, we make use of the precondition technique used in DEFLATED-GMRES($m$, $k$) method in this section.

It has been observed that the convergence of the restarted version of

$\epsilon$ is the tolerance for the residual norm;
$i := 0$; $D_{pre} := -1$;
$convergence := false$;
choose $\boldsymbol{x}_0$;
**until** $convergence$ **do**
  $\boldsymbol{r}_0 := \boldsymbol{b} - A\boldsymbol{x}_0$; $\beta := \| \boldsymbol{r}_0 \|_2$; $\boldsymbol{v}_1 := \boldsymbol{r}_0/\beta$;
  **for** $m = 1$**;** $m \leq m_{\max}$ **do**
    $i := i + 1$;
    $\bar{\boldsymbol{v}} := A\boldsymbol{v}_m$;
    **for** $j = 1$**,** $2$**,** $\ldots$**,** $m - 1$ **do**
      $\bar{H}(j, m) := \boldsymbol{v}^H \bar{\boldsymbol{v}}$;
      $\bar{\boldsymbol{v}} := \bar{\boldsymbol{v}} - \bar{H}(j, m)\boldsymbol{v}_i$;
    **endfor**
    $\bar{H}(m + 1, m) := \| \bar{\boldsymbol{v}} \|_2$;
    $\boldsymbol{v}_{m+1} := \bar{\boldsymbol{v}}/ \| \bar{\boldsymbol{v}} \|_2$;
    $H(i, j) := \bar{H}(i, j)$; $(i, j = 1, \ldots, m)$
    **if** $\| \boldsymbol{r}_m \|_2 < \epsilon$ **then**
      $convergence := true$;
      **exit;**
    **endif;**
    compute $\boldsymbol{f}_m = H^{-H}\boldsymbol{e}_m$ and $\mathcal{H} = H + h^2_{m+1,m}\boldsymbol{f}_m\boldsymbol{e}^H_m$;
    compute the max eigenvalue $\lambda_{\max}$ of $H$;
    compute the max eigenvalue $\bar{\lambda}_{\max}$ of $\mathcal{H}$;
    $D_{cur} := |\lambda_{\max} - \bar{\lambda}_{\max}|$;
    **if** $i > 1$ **then**
      **if** ($D_{cur} > D_{pre}$ **and** $m \geq m_{\min}$) **or** $m = m_{\max}$ **then**
        $\boldsymbol{y} = \min_y \| \beta\boldsymbol{e}_1 - \bar{H}\boldsymbol{y} \|_2$;
        $\boldsymbol{x}_m = \boldsymbol{x}_0 + V_m\boldsymbol{y}, \boldsymbol{r}_m = b - A\boldsymbol{x}_m$;
        $\boldsymbol{x}_0 := \boldsymbol{x}_m$; $D_{pre} := D_{cur}$;
        **break;**
      **endif;**
    **endif;**
    $D_{pre} := D_{cur}$; $m := m + 1$;
  **endfor**

Figure 1: **Algorithm** RITZ-GMRES($m_{\min}$, $m_{\max}$)

GMRES method may be slower than the full GMRES method. It appears as if the restarting procedure loses the information on the smallest Ritz values [3, 4]. In order to overcome this disadvantage, a preconditioning technique, which named deflation, is used in DEFLATED-GMRES($m$, $k$) method [3, 4].

Let $|\lambda_1| \leq |\lambda_2| \leq \cdots \leq |\lambda_n|$ be the Ritz values of $A$. $\boldsymbol{u}_i$ is the Ritz vector respect to $\lambda_i$. In DEFLATED-GMRES($m$, $k$) method, a fixed number $l$ ($= 1$ in this paper) of Ritz vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_l$ are pulled out after each restart. They are used to increase an orthonormal matrix $U_j$. Preconditioner described in (12) is updated after each restart until the dimension of $U_j$ equals to a number $k$.

$$M^{-1} = I_n + U_j \left( |\lambda_n| T_j^{-1} - I_j \right) U_j^H, \quad T_j = U_j^H A U_j \tag{12}$$

$\lambda_n$ is the maximum Ritz value of $A$ and $I$ is the identity matrix. After preconditioned by (12), smallest Ritz values $\lambda_1, \ldots, \lambda_j$ are removed.

As mentioned above, DEFLATED-GMRES($m$, $k$) method makes use of the Ritz values and Ritz vectors. Therefore, the extra cost for the application of the new restart strategy is only the cost for computing the maximum harmonic Ritz value.

Now we apply the restart strategy proposed in section 3 to DEFLATED-GMRES($m$, $k$) method. Note that when the dimension of Krylov subspace is too small, the computed Ritz vectors are not reliable. We suggest that the adaptive restarting scheme should not be executed until no Ritz vector is needed any more, in other words, until the dimension of $U_j$ equals to $k$:

Run DEFLATED-GMRES($m_{\max}$, $k$) for the first $k/f$ cycles;

Call RITZ-GMRES($m_{\min}$, $m_{\max}$) with preconditioner $M^{-1}$.

This preconditioned algorithm is denoted as DEFLATED-RITZ($m_{\min}$, $m_{\max}$, $k$), where $k$ denotes the max dimension of orthonormal matrix $U$ in (12).

# 5 Numerical experiments

In this section we provide a few experimental results to show the efficiency of the adaptive restarting scheme proposed in section 3. At first, we compare RITZ-GMRES($m_{\min}$, $m_{\max}$) with classical GMRES($m$) method without preconditioning in example 1, 2. Then, we compare DEFLATED-RITZ($m_{\min}$, $m_{\max}$, $k$) with DEFLATED-GMRES($m$, $k$) mainly in example 3.

All experiments have been performed on Dell PowerEdge 1750 computer (CPU: Intel(R) Xeon(R) 3.00 GHz, OS: Red Hat Linux 9.0, 4GB main memory), used in single processor mode in double precision. In all tests, initial guess $\boldsymbol{x}_0$ is set to zero and the system is scaled so that the initial residual vector has unit length. The tolerance $\epsilon$ for the residual norm is set to

Table 1: Results for example 1 (T: time(sec) I: iterations)

| Dh | $2^{-3}$ | | $2^{-4}$ | | $2^{-5}$ | | $2^{-6}$ | | $2^{-7}$ | |
|---|---|---|---|---|---|---|---|---|---|---|
| Method | I | T | I | T | I | T | I | T | I | T |
| GMRES(10) | (-5.19) | 2039 | (-5.35) | 2035 | (-5.51) | 2042 | (-5.44) | 2041 | (-4.85) | 2037 |
| GMRES(20) | (-6.35) | 3090 | (-6.66) | 3084 | (-7.60) | 3091 | (-7.68) | 3081 | (-6.50) | 3083 |
| GMRES(30) | (-8.66) | 4138 | (-9.63) | 4133 | (-9.08) | 4135 | (-10.11) | 4126 | (-8.47) | 4131 |
| GMRES(40) | (-10.52) | 5189 | (-11.49) | 5190 | 18235 | 4741 | (-11.30) | 5189 | (-11.09) | 5185 |
| GMRES(50) | (-11.98) | 6252 | (-10.06) | 6243 | 11911 | 3711 | 14709 | 4592 | 19340 | 6030 |
| RITZ-GMRES(1,50) | 19305 | <u>1976</u> | 19225 | <u>2693</u> | 18693 | <u>1187</u> | 10170 | <u>1187</u> | 9267 | <u>1163</u> |

Value in ( ) is $\log_{10} ||\boldsymbol{r}_m||_2$ if the method doesn't convergence after 20,000 iterations.

Table 2: Restart cycle $m$ of RITZ-GMRES method for example 1

| $Dh$ | $2^{-3}$ | $2^{-4}$ | $2^{-5}$ | $2^{-6}$ | $2^{-7}$ |
|---|---|---|---|---|---|
| Average value of $m$ | 4.77 | 4.92 | 5.02 | 5.85 | 5.65 |
| Maximum value of $m$ | 25 | 27 | 29 | 31 | 28 |

$10^{-12}$. Execution of each method is interrupted if the residual norm does not converge after 20,000 iterations. $m_{\max}$ is set to 50 and $m_{\min}$ is set to 1 in example 1, 2, and 5 in example 3. CLAPACK's routines dgeev and zgeev are used for the computation of the Ritz and harmonic Ritz values and vectors [2].

[**Example 1**] We consider the problem, which arises from the 5-point center difference discretization of the elliptic partial differential problem in the unit square region $\Omega = [0,1] \times [0,1]$.

$$\begin{aligned} -u_{xx} - u_{yy} + D((y-1/2)u_x(x,y) + (x-1/3)(x-2/3)u_y(x,y))) &= G(x,y), \\ u(x,y)|_{\partial\Omega} &= 1 + xy, \end{aligned}$$

where $G(x,y)$ is defined so that $u(x,y) = 1 + xy$ on $\Omega$. Mesh width $h$ is set to 1/513. Hence, the matrix's dimension is $262,144$. $Dh$ is set to $2^{-3}$, $2^{-4}$, $2^{-5}$, $2^{-6}$, $2^{-7}$. Table 1 presents the results for various choices of $m$ and $Dh$. For each $Dh$, the shortest computation time is underlined. From Table 1, we see that RITZ-GMRES(1, 50) is much more successful than the classical GMRES($m$) method since it converges for all the $Dh$. The computation time is much shorter too.

Next we study the real restart cycles of RITZ-GMRES(1, 50). Average and maximum values of the real restart cycles are shown in Table 2. From Table 2 we see that the real cycles are rather small for all $Dh$. As an example, we plot all the restart cycles in Figure 2 for $Dh = 2^{-4}$. Since an average of the real restart cycles is about 5 and maximum is 27 for $Dh = 2^{-4}$, we compare RITZ-GMRES(1, 50) to GMRES(5) and GMRES(27) for $Dh = 2^{-4}$. Their convergence is shown in Figure 3, 4. From Figure 3, 4, we find out that RITZ-

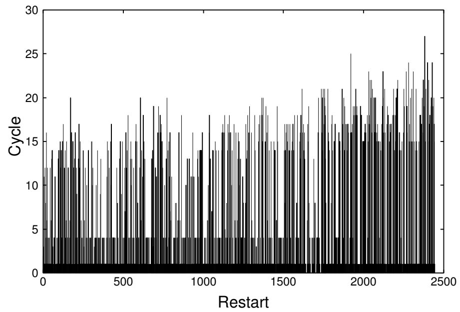


Figure 2: Restart cycle of RITZ-GMRES method for example 1 ($Dh = 2^{-4}$)

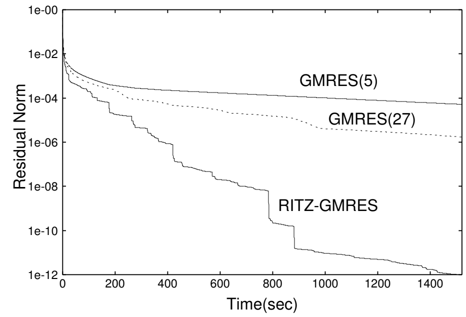


Figure 3: Residual norm vs. computation time for example 1 ($Dh = 2^{-4}$)

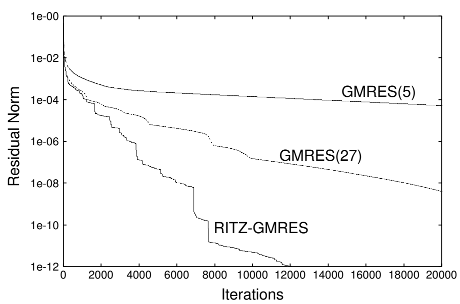


Figure 4: Residual norm vs. iterations for example 1 ($Dh = 2^{-4}$)

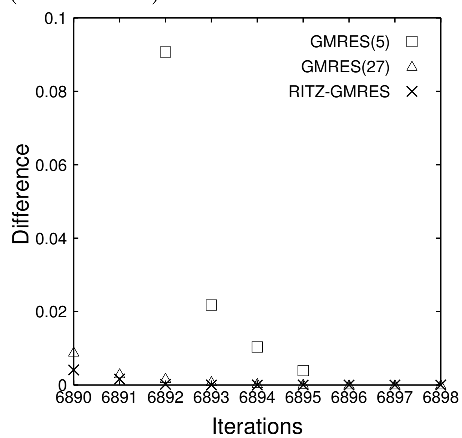


Figure 5: Distribution of $|\lambda_{\max} - \bar{\lambda}_{\max}|$ of RITZ-GMRES method for example 1 ($Dh = 2^{-4}$)

GMRES(1, 50) method not only requires less memory and computation time per cycle, but also works well. These results show that RITZ-GMRES($m_{\min}$, $m_{\max}$) method is much more efficient than the classical GMRES($m$) method by using the new restart strategy.

From Figure 4, we see that the residual norm of RITZ-GMRES ($m_{\min}$, $m_{\max}$) method converges much faster than the other two methods nearby step 6897. Then, we plot the values of $|\lambda_{\max} - \bar{\lambda}_{\max}|$ nearby step 6897 in Figure 5. From Figure 5 we see that the values of $|\lambda_{\max} - \bar{\lambda}_{\max}|$ in RITZ-GMRES($m_{\min}$, $m_{\max}$) method are smaller than those in GMRES(5) and GMRES(27) method. This result coincides with the matter that the value of $|\lambda_{\max} - \bar{\lambda}_{\max}|$ can be used to evaluate the stagnation of the residual norm of GMRES method.

Table 3: Results for example 2

| | GMRES(10) | GMRES(20) | GMRES(30) | GMRES(40) | GMRES(50) | RITZ-GMRES |
|---|---|---|---|---|---|---|
| Iterations | — | 7868 | 7588 | 5614 | 3187 | 5951 |
| Time(sec) | — | 84 | 102 | 92 | 61 | 36 |

—: not convergence after 20,000 iterations.

Table 4: Results for example 3

| | GMRES(10) | GMRES(20) | GMRES(30) | GMRES(40) | GMRES(50) |
|---|---|---|---|---|---|
| Iterations | 18619 | 9430 | 6419 | 4947 | 4088 |
| Time(sec) | 680 | 558 | 530 | 524 | 540 |

[**Example 2**] Next, we consider a linear system MEMPLUS from Matrix Market [8], which has different structure from example 1. Its coefficient matrix is a real nonsymmetric matrix of dimension 17,758, which has 126,150 nonzero entries. From Table 3 we know that RITZ-GMRES(1, 50) method still works better than the classical GMRES($m$) method. The average of restart cycles in RITZ-GMRES(1, 50) is 3.21, maximum is 16. The comparison of RITZ-GMRES(1, 50) with GMRES(4) and GMRES(16) is not described here since the results are similar to example 1.

[**Example 3**] At last, we mainly compare DEFLATED-GMRES($m$, $k$) to DEFLATED-RITZ($m_{\min}$, $m_{\max}$, $k$) method. The matrix is a bidiagonal matrix with entries $1+i$, $2+2i$, $\ldots$, $16384+16384i$ on the main diagonal and $0.1+0.1i$ on the super diagonal. The right-hand side has all entries $1.0+1.0i$.

At first, we show the results of classical GMRES($m$) in Table 4. The results of DEFLATED-GMRES($m$, $k$) and DEFLATED-RITZ($m_{\min}$, $m_{\max}$, $k$) for variable $k$ are shown in Table 5. In DEFLATED-GMRES($m$, $k$) and DEFLATED-RITZ($m_{\min}$, $m_{\max}$, $k$) method, the parameter $m$ is fixed to 50 since the Ritz vectors used in preconditioner (12) are not reliable when $m$ is too small. Comparing Table 4 to Table 5, we see that both DEFLATED-GMRES($m$, $k$) and DEFLATED-RITZ($m_{\min}$, $m_{\max}$, $k$) method work better than the classical GMRES($m$) method because of preconditioning. From Table 5, we also know that DEFLATED-RITZ($m_{\min}, m_{\max}, k$) is more successful than DEFLATED-GMRES($m$, $k$). These results show that the new restart

Table 5: Results for example 3 (T: computation time(sec) I: iterations)

| Method | $k=1$ | | $k=2$ | | $k=3$ | | $k=4$ | |
|---|---|---|---|---|---|---|---|---|
| | I | T | I | T | I | T | I | T |
| DEFLATED-GMRES(50,$k$) | 3203 | 414 | 3057 | 416 | 2681 | 380 | 2313 | 344 |
| DEFLATED-RITZ(5,50,$k$) | 5058 | 393 | 3363 | 269 | 3676 | 293 | 2887 | 236 |

strategy still works well in preconditioned GMRES($m$) method.

# 6 Concluding Remark

In this paper, we propose and study new adaptive restart strategy of GMRES($m$) algorithm with using the difference of the Ritz and harmonic Ritz values. The difference is estimated by the absolute value of the difference of the maximum Ritz value and the maximum harmonic Ritz value. Numerical experiments show that the proposed algorithm can be much better than the classical GMRES($m$) method with fixed restart cycle. We also apply this restart strategy to DEFLATED-GMRES($m$, $k$) as an example.

Further research is needed to study the stability of the proposed restart strategy, including application in other preconditioned GMRES($m$) methods, for solving general non-Hermitian linear systems.

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