On the distributions of maximum downfalls of a Brownian motion with drift

Albert N. Shiryaev

Steklov Institute of Mathematics, 8 Gubkina st., Moscow 119991, Russia
[albertsh@mi.ras.ru]

Let $B = (B_t)_{t\geq 0}$ be a Brownian motion with drift ($B_t = \mu t + W_t$, where $W = (W_t)_{t\geq 0}$ is a standard Wiener process). We consider the problem of finding distributions of the following characteristics which show how big are drops "from a peak to a bottom" of the trajectories of B on time interval $[0, T]$:

$$D_T = \max_{0\leq s\leq s'\leq T}(B_s - B_{s'})$$

(the largest drop from a peak to a bottom),

$$\overline{D}_T = B_{\sigma_T} - \min_{\sigma_T\leq s'\leq T} B_{s'}$$

(the largest drop from the absolute maximum B_{σ_T}, where $\sigma_T = \min\{0 \leq s \leq T : B_s = \max_{0\leq u\leq T} B_u\}$ to the (partial) minimum on interval $(\sigma_T, T]$),

$$\underline{D}_T = \max_{0\leq s'\leq \sigma'_T} B_{s'} - B_{\sigma'_T}$$

(the largest drop from the (partial) maximum on interval $[0, \sigma'_T)$ to the absolute minimum $B_{\sigma'_T}$, where $\sigma'_T = \min\{0 \leq s \leq T : B_s = \min_{0\leq u\leq T} B_u\}$).

We prove, in particular, that

(a) $D_T = \max(\overline{D}_T, \underline{D}_T)$, $\overline{D}_T \overset{\text{law}}{=} \underline{D}_T$;

(b) $D_T \overset{\text{law}}{=} \max_{0\leq s'\leq T} |X_{s'}|$;

(c) $\overline{D}_T \overset{\text{law}}{=} \max_{g_T\leq s'\leq T} |X_{s'}|$,

where $X = (X_t)_{t\geq 0}$ is a "bang-bang" process,

$$dX_t = -\mu \operatorname{sgn} X_t\, dt + dW_t, \qquad X_0 = 0,$$

and g_T is the last zero of B before time T.

We present also results about explicit formulae for distributions of D_T and $\overline{D}_T$ for the case $\mu = 0$ and give a double Laplace transform for the case $\mu \neq 0$.