# On tail distributions of supremum and quadratic variation of cádlág local martingales

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In the case of a continuous local martingale there have been works by Azema, Gundy, and Yor[1], Elworthy, Li, and Yor[2], and Takaoka[4] etc. Recently, by Liptser and Novikov[3] they were extended to the case of a local martingale with uniformly bounded jumps. We introduce the main result in that paper;

**Theorem 0.1** *Let $M = \{M_t\}_{t\in\mathbf{R}_+}$ be a locally square integrable cádlág martingale defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbf{R}_+}, P)$ the filtered probability space with standard general conditions. Assume that $\langle M\rangle_\infty = \lim_{t\to\infty}\langle M\rangle_t < \infty$ a.s and $\{M_\tau^+\}_{\tau\in\mathcal{T}}$ is uniformly integrable, where $\mathcal{T}$ is the set of stopping times $\tau$. Then*

(*i*) $0 \le E[M_\infty] \le E[M_\infty^+] < \infty$.

*Besides,*

(*ii*) *if $\{\Delta M_\tau\}_{\tau\in\mathcal{T}}$ is uniformly integrable, then*

$$\lim_{\lambda\to\infty}\lambda P(\sup_{t\in\mathbf{R}_+}(M_t^-) > \lambda) = E[M_\infty];$$

(*iii*) *if $|\Delta M| \le K$ and $E[e^{\epsilon M_\infty}] < \infty$ for some $K > 0$ and $\epsilon$, then*

$$\lim_{\lambda\to\infty}\lambda P(\sqrt{\langle M\rangle_\infty} > \lambda) = \lim_{\lambda\to\infty}\lambda P(\sqrt{[M]_\infty} > \lambda) = \sqrt{\frac{2}{\pi}}E[M_\infty].$$

In this presentation, we will present the result without the uniform boundedness assumption for jumps. But, to obtain the characterization of a tail distribution of quadratic variation of a local martingale $M$, we replace it by another assumptions: "the quasi left-continuity of $M$ and the exponential moment in terms of the compensator of the counting measure of $\Delta M$."

1. Azema, Gundy, Yor (1980). Sur lintegrabilite uniforme des martingales continues. *Seminaire de Probabilites XIV, LNM 784*, Springer, pp.249-304.
2. Elworthy, Li, Yor (1997). On the tails of the supremum and the quadratic variation of strictly local martingales. *Seminaire de Probabilites XXXI, LNM 1655*, Springer, pp.113-125.
3. Liptser, Novikov (2004). On tail distributions of supremum and quadratic variation of local martingales. preprint.
4. Takaoka (1999). Some remarks on the uniform integrability of continuous martingales. *Seminaire de Probabilites XXXIII, LNM 1709*, Springer, pp.327-333.