Using Bellman's Principle without the Bellman Equation: New Parallel-Computing-Capable Numerical Methods for Optimal Stopping, with Applications to Financial Economics in View

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Many optimal stopping problems – especially the ones encountered in financial economics – essentially come down to constructing a function $(t,x) \longrightarrow u(t,x)$ that has the property:

$$u(t,x) = \max[\Lambda(x), e^{-r\,\delta}\mathbf{E}[u(t+\delta, X_\delta^x)]] + o(\delta)\,, \tag{1}$$

where $\Lambda(\cdot)$ is the termination payoff function and $(X_t^x)_{t\geq 0}$ is some Markov process governed by some stochastic equation of the form

$$X_t^x = x + \int_0^t \sigma(X_s)dW_s + \int_0^t a(X_s)ds\,. \tag{2}$$

Traditionally, the function $u(\cdot,\cdot)$ has been computed by way of solving the associated Bellman equation, for which various numerical techniques – mostly variations of the finite difference scheme – have been developed. A new approach, which takes advantage of the recent developments in computing technology and allows one to construct the function $u(\cdot,\cdot)$ directly, i.e., without any reference to the Bellman equation, by way of backward induction governed by Bellman's principle (1), is described in [1]. In this approach, equation (2) is approximated by an equation with affine coefficients which admits an "explicit" solution in terms of integrals of the exponential Brownian motion. The expectation in the right side of (1) is calculated in [1] by using a rather crude approximation of the distribution of the *integral of the exponential Brownian motion* (IEBM). In this paper various methods for computing integrals involving the probability density function of the IEBM will be discussed. While our main interest in such calculations is motivated by optimal stopping problems and the general procedure described in [1], the distribution of the IEBM has been of particular interest in mathematical finance in connection with the so called Asian options, as explained in [2] and [3]. New methods for computing the function $u(\cdot,\cdot)$ directly from Bellman's principle (1) will be presented and the parallel computing aspects of such procedures will be discussed.

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2. Yor, M. (1992). On Some Exponential Functionals of Brownian Motion. *Adv. Appl. Prob.*, **24**, 509–531.
3. Yor, M. (2001). Exponential Functionals of Brownian Motion and Related Processes. Springer-Verlag, Berlin.